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Considerations of the stability of certain heterogeneous shear flows including some inflexion-free profiles

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We use some well-known properties of the Taylor–Goldstein equation to generate a set of stably stratified shear flows with known singular neutral-mode solutions. The novel feature of the analysis is that it includes such solutions for flows in which, proceeding upwards from a rigid boundary, the Brunt–Väisälä frequency and the flow shear do not change sign and are monotonically decreasing functions of height. Such profiles are much closer to the conditions met in work on the atmospheric boundary layer than the more frequently used inflected flow profiles.

1. Introduction

The dynamic stability of stably stratified shear flows is a well-studied topic. The general theory as it exists today is to be found in the papers by Miles (1961, 1963) and Howard (1961). Some further insight into the problem is gained through the considerations of the energetics by Ludlam (1967) and Hines (1971), while laboratory confirmation of the theory is obtained from the work of Thorpe (1968) and others. No general sufficient condition for instability of these flows has been obtained, although we have the results of a number of calculations for particular flow profiles to guide us. The work in this note was undertaken to eliminate suspicion that flow profiles relevant to the atmospheric boundary layer would obey Rayleigh's (1880) inflexion theorem and thus be stable unless subjected to finite disturbances that significantly altered the mean flow.

The situation may be stated as follows. Rayleigh's flex-point theorem is applicable to shear flows in which gravitational stratification is inoperative. A more general law reducing to Rayleigh's theorem in the appropriate limit has been obtained (e.g. Miles 1961) but combined with the so-called semi-circle theorem it appears to be trivially obeyed by all potentially unstable modes and sets no restrictions on the flex points of the stably stratified system. Arguing on the 'physical' grounds that stable stratification should increase the requirements to be met before the breakdown of shear flows, Kuo (1963) suggests that the simple Rayleigh theorem should be operative. This ignores the fact that the introduction of the stable stratification allows entirely new modes of wave motion (the internal gravity waves). In fact we know of particular examples (e.g. Thorpe 1969) in which stratification can destabilize a flow profile.

However, the stratified flow profiles that have been found to allow instabilities (as in Taylor 1931; Goldstein 1931; Drazin 1958; Menkes 1961; Thorpe 1969; Hazel 1972; Miles & Howard 1964; Jones 1968) all contain inflexions in the flow profile, while the constant-shear flow (Taylor 1931; Case 1960) and a piecewise linear profile corresponding in some sense to an inflexion-free 'real' flow (Jones 1968) show a complete absence of instability. Likewise Miles (1967) considered a semi-infinite inflexion-free jet as a model of atmospheric and oceanic boundary-layer flows, and found it to be stable against infinitesimal perturbations. He remarks on the inflexion-free nature of the profile as being of possible significance in this respect. The density stratification of Miles' model, in which the Brunt–Väisälä frequency tends towards zero in the upper space, eliminates the possibility of having an outgoing internal gravity wave at the upper boundary (infinity). In this respect at least the model is closer to the unstratified Rayleigh-type flows than are many boundary-layer conditions encountered.

As we shall now show, it is possible to manufacture inflexion-free velocity profiles which, together with stratification like that in the atmospheric boundary layer, allow singular neutral modes. The work of Miles (1963) then implies the existence of contiguous exponentially unstable modes.

2. Formulation

Take a rectangular x, z co-ordinate system with $\hat{\mathbf{z}}$ in the direction opposite to that of the gravitational acceleration $\mathbf{g}$. Let the unperturbed flow field be given by some velocity profile $\hat{\mathbf{x}}U(z)$, and through the density field $\rho(z)$ define the stability parameter n known as the Brunt–Väisälä frequency:

$$n^2 = -g\rho'/\rho. \tag{1}$$

A prime represents a derivative with respect to z. The form chosen in (1) shows that we are confining ourselves to the study of incompressible flows. We shall further restrict consideration to perturbations with vertical scales that allow the Boussinesq approximation.

Then flow perturbations of the form

$$\psi = \phi(z)\, e^{ik(Ct-x)} \tag{2}$$

in the vertical fluid velocity (say) are governed by the equation

$$\frac{d^2\phi}{dz^2} + \left[\frac{n^2}{(U-C)^2} - k^2 - \frac{U''}{U-C}\right]\phi = 0, \tag{3}$$

together with appropriate boundary conditions.

The work of Miles (1961) shows that any unstable modes ($\operatorname{Im} C \neq 0$) and the singular neutral modes implied by their existence have certain properties, among which we find the following of interest.

(i) $U(z)_{\min} < \operatorname{Re} C < U(z)_{\max}$, so that $\operatorname{Re}[U(z)-C] = 0$ at some point(s) $z = z_c$ (say).

(ii) At at least one such point z_c it is necessary that

$$R_i(z_c) \equiv n^2(z_c)/U^2(z_c) \leqslant \tfrac{1}{4}. \tag{4}$$

(iii) About such a point z_c the solutions $\phi(z)$ for the singular neutral modes are of the form

$$\phi(z-z_c) = A_+(z-z_c)^{\frac{1}{2}+\nu}Y_+(z-z_c)+A_-(z-z_c)^{\frac{1}{2}-\nu}Y_-(z-z_c), \tag{5}$$

where

$$\nu = (\tfrac{1}{4}-R_i(z_c))^{\frac{1}{2}} \tag{6}$$

and Y_+ and Y_- are real analytic functions in the neighbourhood of z_c.

Now let U and n^2 be single-valued analytic functions throughout the z interval on which solutions are required, so that there is only one singular point z_c for any given mode. Then the solutions $\phi^+ = (z-z_c)^{\frac{1}{2}+\nu}Y_+$ and $\phi^- = (z-z_c)^{\frac{1}{2}-\nu}Y_-$ and their analytic continuations over the interval are real functions of z to within a phase factor which may change only across the point z_c. From the reality of these solutions we may easily obtain classes of boundary-value problems for which singular neutral modes exist.

2.1. *Inflected flow profiles between rigid boundaries*

Select an n^2 and U, monotonic functions of z on the real interval $I = (-\infty, \infty)$, that satisfy the conditions

$$\left.\begin{aligned} &R_i < \tfrac{1}{4} \text{ at at least one point } z_p \text{ of } I,\\ &\frac{n^2}{(U-U(z_p))^2}-\frac{U''}{U-U(z_p)} > P^2 \quad \text{for all} \quad |z| > Q^2 \quad \text{on} \quad I, \end{aligned}\right\} \tag{7}$$

where P and Q are real numbers.

It is very simple to construct examples of such profiles if U is allowed to have a flex point. Then restrict k^2 to some value such that

$$k^2 < P^2. \tag{8}$$

Identify C with $U(z_p)$ (so that $z_c = z_p$) and select a solution about z_c of the form (5) with $A_- \equiv 0$. Then, since in each of the intervals (Q^2, ∞) and $(-\infty, -Q^2)$ the analytic continuation of this solution is a real function of z (times some phase factor), conditions (7) and (8) require it to have zeros spaced at intervals of z less than $\pi/(P^2-k^2)^{\frac{1}{2}}$, as may be inferred from Sturm's fundamental theorem.

Arbitrarily selecting one such zero in each of the upper and lower intervals as a location for a rigid boundary, we have constructed a flow system for which this solution is an acceptable singular neutral mode.

A similar treatment based on the solution (5) with A_+ taken to be identically zero would yield a similar result with differently located boundaries.

2.2. *Inflexion-free profiles between rigid boundaries*

None of the steps followed in § 2.1 depended on U having a flex point. As long as the profile of n^2 and U satisfy the inequalities (7), the argument carries over.

If we take a flow profile in which U'' tends towards zero at large z, where U approaches a constant from below and n^2 a constant from above, then (7) is trivially obeyed for $z > z_p$. However, if U'' is not to change sign U^2 must be unbounded as $z \to -\infty$. Hence, if (7) is to be obeyed on the lower subinterval of I, n^2 must also increase at at least the same rate.

A specific example of an inflexion-free profile obeying these conditions is given by

$$U = U(\infty) + \tfrac{1}{2}\ln(1+e^{2z}) - z, \tag{9}$$

$$n^2 = n^2(\infty) + B^2[\tfrac{1}{2}\ln(1+e^{2z}) - z]^2 \tag{10}$$

in a suitable non-dimensional form.

Clearly

$$\lim_{|z|\to\infty} \{U''/(U-C)\} = 0, \tag{11}$$

$$\lim_{z\to\infty} \{n^2/(U-C)^2 = n^2_\infty/(U(\infty)-C)^2, \tag{12}$$

$$\lim_{z\to-\infty} \{n^2/(U-C)^2\} = B^2 \tag{13}$$

and we may follow the procedure in § 2.1 to construct profiles allowing singular neutral modes. It would appear that many other suitable profiles can be formed, demonstrating conclusively that Rayleigh's flex-point theorem does not apply to the general stratified shear flow.

In this work we have made use of the reality of ν and the non-zero values of n^2 at the boundaries. If either of these conditions is lost, as in the transition to flows with Richardson numbers greater than $\frac{1}{4}$ throughout, or to flows where the stratification is lost over any infinite subinterval of I, the demonstration fails.

3. Relation to Yih's sufficient conditions for the stability of stratified flows

Yih (1970) has obtained joint conditions on the velocity and density profiles that provide for stability of the system (see theorem 3 of his paper). These results are the most powerful extension of Rayleigh's theorem known to the author.

As must be expected, the profiles obtained in §§ 2.1 and 2.2 above do not meet Yih's conditions. It is interesting to note that the construction in § 2.2 produces a stratification that increases in the direction of increasing shear. It might be thought that such a system would be more stable than one in which the greater shears acted in the regions of lesser static stability, but in fact, comparison with Yih's results shows the contrary.

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